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GROUPS ISOMORPHIC TO ALL THEIR NON-TRIVIAL NORMAL SUBGROUPS*

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ABSTRACT

In answer to a question of P. Hall, we supply another construction of a group which is isomorphic to each of its non-trivial normal subgroups.

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1. Introduction

In the early 1970's, the following question was asked by Philip Hall of J.C. Lennox (see [4]) and appeared later in [3] in the form:

> Must a non-trivial group, which is isomorphic to each of its non-trivial normal subgroups, be either free of infinite rank, simple or infinite cyclic?

This was answered in the negative by Obraztsov in [6], where he used the technique of graded diagrams developed by Ol'shanskii in [7].

In this note we provide a different construction of a non-trivial group which is isomorphic to each of its non-trivial normal subgroups. This construction makes special use of the notion of homogeneous and universal structures in model theory and well-known facts about HNN extensions and free products with amalgamation. In our view, the following construction yields a transparent proof of the existence of such groups and shows how basic ideas from model theory can be used to resolve questions in other areas of mathematics. While Obraztsov's proof is based on the deep theory of the geometry of defining relations in groups, our approach is accessible to a graduate student with basic knowledge of model theory and group theory. Our main result is the following:

MAIN THEOREM: *Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$. Then there exists a group G of cardinality κ with a descending principal series*

$$G \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

such that $\bigcap_{n<\omega} G_n = 1$, $G_n \cong G$ for each $n < \omega$, and any normal subgroup of G is equal to some G_n.

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2. Homogeneous and universal structures

In this section we consider a class of models $\mathcal{M}$ and give a brief summary of its relevant properties. Notation and terminology observed here follow that in [1].

Let $\mathcal{M}$ be the collection of relational structures in the language L of groups with constant symbol 1 (whose value in any model will be the group identity), the axioms of groups and an additional sequence P_n ($n \in \omega$) of unary predicates having the following properties: If A is a group and P_nA denotes P_n applied to

A, then P_nA is a normal subgroup of A (denoted $P_nA \lhd A$) and $P_{n+1}A \subseteq P_nA$. In terms of first order language, this is to say that:

(1) $1 \in P_nA$.

(2) If $x, y \in P_nA$, then $xy^{-1} \in P_nA$.

(3) If $x \in P_nA$ and $z \in A$, then $z^{-1}xz \in P_nA$.

(4) If $x \in P_{n+1}A$, then $x \in P_nA$.

Thus the class $\mathcal{M}$ of all such L-models is defined to be

$$\{(P_nA)_{n<\omega}\colon P_nA \text{ is a group }, P_nA \lhd P_0A \text{ and } P_{n+1}A \subseteq P_nA \ (n < \omega)\}.$$

To simplify notation, we let $P_0A = A$ and abbreviate the sequence $(P_nA)_{n<\omega}$ of groups as (P_nA).

Let (P_nA), $(P_nB) \in \mathcal{M}$. The relational structure (P_nA) is said to be a **substructure** of (P_nB) (equivalently, (P_nB) is an **extension** of (P_nA)) if $P_nA = P_nB \cap P_0A$ for all $n < \omega$. We denote this by $(P_nA) \leq (P_nB)$. Let $\varphi\colon P_0A \to P_0B$ be a map. We say that φ is a **homomorphism** of (P_nA) into (P_nB) if φ is a group homomorphism and $\varphi(P_nA) = P_nB \cap \varphi(P_0A)$ $(n < \omega)$. A homomorphism φ is called an **embedding** of (P_nA) into (P_nB) if φ is one-to-one. Hence if $(P_nA) \leq (P_nB)$, then the identity map *id*: $(P_nA) \to (P_nB)$ is an embedding. If φ is one-to-one and onto, then φ is called an **isomorphism** and (P_nA) is said to be **isomorphic** to (P_nB). This we denote by $(P_nA) \cong (P_nB)$. We refer to the cardinality of P_0A as the **cardinality of** (P_nA).

PROPOSITION 2.1: *The following properties hold in $\mathcal{M}$:*

(I) *$\mathcal{M}$ contains structures of arbitrarily large cardinality.*

(II) *If $(P_nA) \in \mathcal{M}$ and $(P_nA) \cong (P_nB)$, then $(P_nB) \in \mathcal{M}$.*

(III) *If (P_nA_1), $(P_nA_2) \in \mathcal{M}$, there exist $(P_nC) \in \mathcal{M}$ and embeddings*

$$\varphi_i\colon (P_nA_i) \hookrightarrow (P_nC) \quad (i = 1, 2).$$

(IV) (Amalgamation Property) *Let (P_nA), (P_nB_1), $(P_nB_2) \in \mathcal{M}$ such that*

$$\varphi_i\colon (P_nA) \hookrightarrow (P_nB_i) \quad (i = 1, 2)$$

are embeddings. Then there is some $(P_nC) \in \mathcal{M}$ and embeddings

$$\gamma_i\colon (P_nB_i) \hookrightarrow (P_nC) \quad (i = 1, 2) \quad \text{such that } \gamma_1 \circ \varphi_1 = \gamma_2 \circ \varphi_2.$$

(V) *The union of any chain of structures $(P_nA) \in \mathcal{M}$ is still in $\mathcal{M}$.*

(VI_κ) *Let κ be an uncountable cardinal. If $(P_nA) \in \mathcal{M}$ and $(P_nC) \le (P_nA)$ such that (P_nC) is of cardinality $< \kappa$, then there exists $(P_nB) \in \mathcal{M}$ of cardinality $< \kappa$ such that $(P_nC) \le (P_nB) \le (P_nA)$.*

Proof: (II) This is clear by the definition of a homomorphism between elements of $\mathcal{M}$. (V) is obvious.

(I) Let α be an infinite cardinal. Take any group H of cardinality α and let $P_nA = H$ for all n. Then $(P_nA) \in \mathcal{M}$ with cardinality α.

(III) Take, for instance, $P_nC = P_nA_1 \oplus P_nA_2$ $(n < \omega)$ and the natural injections $\varphi_i\colon P_0A_i \to P_0C$ $(i = 1, 2)$.

(IV) Consider the free product with amalgamation

$$P_0C = P_0B_1 *_{P_0A} P_0B_2$$

and the identity map $id_i\colon P_0B_i \to P_0C$. Then $id_1 \circ \varphi_1 = id_2 \circ \varphi_2$, since $\varphi_1(a) = \varphi_2(a)$ $(a \in P_0A)$ when viewed as elements of P_0C. Define $P_nC = \langle P_nB_1, P_nB_2\rangle^{P_0C}$, the normal subgroup generated by $\langle P_nB_1, P_nB_2\rangle$ in P_0C. Clearly $(P_nC) \in \mathcal{M}$. It suffices to verify that $P_nC \cap P_0B_i = P_nB_i$ $(n < \omega)$. Let

$$G_n^* = P_0B_1/P_nB_1 *_{P_0A/P_nA} P_0B_2/P_nB_2$$

be a free product with amalgamation and

$$\bar{\varphi_{ni}}\colon P_0A/P_nA \hookrightarrow P_0B_i/P_nB_i \quad (i = 1, 2)$$

be the homomorphisms defined by $\bar{\varphi_{ni}}(aP_nA) = \varphi_i(a)P_nB_i$. Consider the canonical epimorphisms

$$\pi_{ni}\colon P_0B_i \to P_0B_i/P_nB_i \quad (n < \omega, i = 1, 2).$$

Then the map

$$\pi_{n1} \cup \pi_{n2}\colon P_0B_1 \cup P_0B_2 \to G_n^*$$

extends to a homomorphism $\pi\colon P_0C \to G_n^*$ such that $\pi \restriction_{P_0B_i} = \pi_{ni}$. If $x \in P_0B_i \setminus P_nB_i$, then $\pi_{ni}(x) \ne 1$ and so $\pi(x) \ne 1$. If $x \in P_nB_1 \cup P_nB_2$, then $\pi(x) = 1$. Thus $\ker\pi = \langle P_nB_1, P_nB_2\rangle^{P_0C} = P_nC$ and so $P_nC \cap P_0B_i = P_nB_i$ $(i = 1, 2)$. Thus $(P_nB_i) \le (P_nC)$.

(VI_κ) Note that (P_nC) is itself in $\mathcal{M}$, hence take $(P_nB) = (P_nC)$. ∎

The preceding properties (I)–$(\mathrm{VI})_\kappa$ and the countability of the language L show that $\mathcal{M}$ is a κ-class, where κ is an uncountable cardinal. Note that if $\mathcal{M}^n$

is defined to consist of all elements in $\mathcal{M}$ such that the first $n+1$ terms of the sequence are all equal (for a fixed n), then $\mathcal{M}^n$ is still a κ-class.

A structure $(P_nA) \in \mathcal{M}$ is said to be $(\mathcal{M},\kappa)$-**homogeneous** if given any two (P_nB_1), $(P_nB_2) \in \mathcal{M}$ of cardinality $< \kappa$ which are substructures of (P_nA) and an isomorphism φ: $(P_nB_1) \to (P_nB_2)$, then there is an isomorphism γ: $(P_nA) \to (P_nA)$ such that $\gamma\restriction_{(P_nB_1)} = \varphi$. A structure (P_nA) is said to be $\mathcal{M}$-**homogeneous** if (P_nA) is of cardinality κ and $(\mathcal{M},\kappa)$-homogeneous. (P_nA) is said to be $\mathcal{M}$-**universal**, if (P_nA) is of cardinality κ and, given any $(P_nB) \in \mathcal{M}$ of cardinality $< \kappa^+$, there is an embedding of (P_nB) into (P_nA). By Jónsson's Theorem (see [1], p. 213 or [2]), if κ is chosen such that $\kappa = \kappa^{<\kappa}$ (e.g., κ is regular and either κ is a limit beth number or the GCH holds), then $\mathcal{M}$ contains an $\mathcal{M}$-homogeneous, $\mathcal{M}$-universal structure of cardinality κ which is unique up to isomorphism. We denote this structure by (P_nG).

3. Normal subgroups of P_0G

We state and verify here special properties of P_0G and use them to obtain the required group.

Proposition 3.1: *Let (P_nG) be $\mathcal{M}$-homogeneous, $\mathcal{M}$-universal of cardinality κ. Then $P_nG \cong P_0G$ $(n < \omega)$.*

Proof: Let $M = (P_nG)$. Define

$$M^n = (P_nG, P_nG, \ldots, P_nG, P_{n+1}G, P_{n+2}G, \ldots),$$

where the first $n+1$ terms in the sequence are all equal to P_nG and the succeeding terms agree with the corresponding terms in M, i.e., if $k > 1$, the $(n+k)$th term of M^n is the $(n+k)$th term of M. Define

$$\mathcal{M}^n = \{(P_nA) \in \mathcal{M}\colon \text{ the first } n+1 \text{ terms of } (P_nA) \text{ are equal}\},$$

which is a κ-class containing M^n. Since M is $\mathcal{M}$-homogeneous, $\mathcal{M}$-universal and $M^n \leq M$, it follows that M^n is $\mathcal{M}^n$-homogeneous and $\mathcal{M}^n$-universal.

Consider $M^- = (P_nG, P_{n+1}G, \ldots)$. We show that M^- is $\mathcal{M}$-homogeneous and $\mathcal{M}$-universal. For then, by Jónsson's Theorem, M must be isomorphic to M^- and so we have shown that $P_0G \cong P_nG$.

Suppose $X = (P_nA) \in \mathcal{M}$ is of cardinality $< \kappa^+$. Define

$$X^0 = (P_0A, P_0A, \ldots, P_0A, P_0A, P_1A, P_2A, \ldots)$$

to be an element of $\mathcal{M}^n$, where the first n terms are all equal to P_0A and the succeeding terms of the chain X^0 are P_nA ($n < \omega$). Since M^n is $\mathcal{M}^n$-universal, there exists an embedding φ: $X^0 \hookrightarrow M^n$. Hence $\varphi \restriction_X : X \hookrightarrow M^-$ is also an embedding, and so M^- is $\mathcal{M}$-universal.

Let $X = (P_nA)$ and $Y = (P_nB)$ be substructures of M^- and φ: $(P_nA) \to (P_nB)$ be an isomorphism. We show that there exists an automorphism γ of M^- such that $\gamma \restriction_X = \varphi$. As in the preceding paragraph, define the elements X^0 and Y^0, which are substructures of M^n. Then φ: $X^0 \to Y^0$ is also an isomorphism. Since M^n is $\mathcal{M}^n$-homogeneous, there exists an automorphism ρ of M^n such that $\rho \restriction_{X^0} = \varphi$. It is clear that $\gamma = \rho \restriction_{M^-}$ is the required automorphism of M^- which extends φ. Thus we have shown that M^- is $\mathcal{M}$-homogeneous. ∎

THEOREM 3.2: *For all $w \in P_nG \setminus P_{n+1}G$, $\langle w \rangle^{P_0G} = P_nG$.*

Proof: By Proposition 3.1, it is enough to prove that for all $w \in P_0G \setminus P_1G$, $\langle w \rangle^{P_0G} = P_0G$. If $g \in P_1G$, then $g = (gw^{-1})w$ and $gw^{-1} \in P_0G \setminus P_1G$. Thus in order to prove that $\langle w \rangle^{P_0G} = P_0G$, we need only show that $P_0G \setminus P_1G \subseteq \langle w \rangle^{P_0G}$.

Suppose that $w, a \in P_0G \setminus P_1G$. Let $H_1 = \langle w, a \rangle$ and $N_1 = H_1 \cap P_1G$. Let $H_2 = H_1 * \langle x \rangle$ be the free product and let $N_2 = N_1^{H_2}$. Let $b = wxw^{-1}x^{-1}$ and $c = axw^{-1}x^{-1}$. By considering the canonical homomorphism

$$H_2 \to (H_1/N_1) * \langle x \rangle$$

we see that

(i) $N_2 \cap H_1 = N_1$,

(ii) $b^n, c^n \notin N_2$ for all $n > 0$.

Let $H_3 = \langle H_2, t \mid tbt^{-1} = c \rangle$ and let $N_3 = N_2^{H_3}$. Let $\bar{b}, \bar{c}$ be the elements of H_2/N_2 corresponding to b, c. By considering the canonical homomorphism

$$H_3 \to \langle H_2/N_2, t \mid t\bar{b}t^{-1} = \bar{c} \rangle$$

we see that $N_3 \cap H_2 = N_2$. Note that in H_3 we have that

$$twxw^{-1}x^{-1}t^{-1} = axw^{-1}x^{-1}$$

and so $a \in \langle w \rangle^{P_0G}$. ∎

THEOREM 3.3: *Let $I = \bigcap_{n<\omega} P_nG$. If H is a normal subgroup of P_0G such that H is not contained in I, then $H = P_nG$ for some $n < \omega$. Thus every non-trivial normal subgroup of P_0G/I is equal to P_kG/I for some $k < \omega$.*

Proof: Let H be a normal subgroup of P_0G such that H is not contained in I. Then there exists a least n such that $H \not\subseteq P_nG$. So $H \subseteq P_{n-1}G$ and there exists $x \in H \setminus P_nG$. By Theorem 3.2, it follows that $H = P_{n-1}G$. ∎

Let F_κ be the free group of infinite rank κ and define $P_nA = F_\kappa^{(n)}$ to be the nth derived group of F_κ. Then $(P_nA) \in \mathcal{M}$ with cardinality κ. Since (P_nG) is $\mathcal{M}$-universal of cardinality κ, the free generators of F_κ correspond to κ distinct elements of P_0G. Since $\bigcap_{n<\omega} P_nA = 1$, P_0G/I has cardinality κ. The proof of the main theorem will now follow, when we take $G = P_0G/I$ and $G_n = P_nG/I$ $(0 < n < \omega)$.

References

[1] J. L. Bell and A. B. Slomson, *Models and Ultraproducts*, North-Holland, Amsterdam, 1974.

[2] M. Droste and R. Göbel, *A categorical theorem on universal objects and its application in abelian group theory and computer science*, Contemporary Mathematics **131** (1992), 49–73.

[3] E. I. Khukhro and V. D. Mazurov, *Unsolved Problems in Group Theory; The Kourovka Notebook*, Russian Academy of Science, Novosibirsk, 1992.

[4] J. C. Lennox, H. Smith and J. Wiegold, *A problem about normal subgroups*, Journal of Pure and Applied Algebra **88** (1993), 169–171.

[5] R. Lyndon and P. Schupp, *Combinatorial Group Theory*, Springer, Berlin, 1977.

[6] V. N. Obraztsov, *On a problem of P. Hall about groups isomorphic to all their non-trivial normal subgroups*, Proceedings of the London Mathematical Society **75** (1997), 79–98.

[7] A. Yu. Ol'shanskii, *Geometry of Defining Relations in Groups*, North-Holland, Amsterdam, 1991.